Anisotropic distribution functions for relativistic galactic nuclei

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(Received 6 September 1995)

An integral transform is derived which, for a system of stars characterized as a spherical equilibrium solution to the collisionless Boltzmann equation of general relativity, allows the reconstruction of a two-integral distribution function from moments of that distribution. Specifically, given a knowledge of the energy density $\rho(r)$ and some constraint on the radial and tangential pressures $P_R(r)$ and $P_T(r)$, a two-integral distribution $f_0(E,J^2)$ consistent with these moments can be computed as a contour integral in the complex plane. This prescription constitutes a straightforward generalization of a transform for the corresponding Newtonian systems developed recently by Qian and Hunter. It also generalizes earlier work by Fackerell for relativistic systems characterized by a one-integral distribution function, $f_0(E)$. This transform has potential applications for the modeling of dense galaxy cusps surrounding supermassive black holes, an enterprise that has recently assumed increased importance in view of high resolution photometry facilitated by the Hubble Space Telescope.

PACS number(s): 05.20.Dd, 04.20.Jb, 98.10.+z

I. INTRODUCTION AND MOTIVATION

In a variety of different contexts, astrophysicists are interested in modeling systems of stars or other objects as solutions to the so-called collisionless Boltzmann equation [1]. In the context of Newtonian mechanics this entails a consideration of the gravitational Vlasov-Poisson system, which tracks the evolution of a one-particle distribution function f in a gravitational potential Φ, the form of which is determined self-consistently by f itself. Alternatively, in the context of general relativity this entails a consideration of the Vlasov-Einstein system, which tracks the evolution of a covariant one-particle f in a space time (M,g_{ab}), with an Einstein tensor $G_a^b[g]$ determined self-consistently by the stress-energy tensor T_a^b associated with f.

The Vlasov-Poisson system is the starting point for much work in galactic dynamics and cosmology. However, under certain circumstances the system of interest may be so dense, and/or the velocities of the objects in question so large, that the Newtonian description must be superceded by a fully relativistic treatment. Thus, in particular, over the years substantial attention has focused on models of ultradense galactic nuclei [2], both as objects in their own right and as potential progenitors for supermassive black holes.

It is well known, both Newtonianly and in general relativity [1], that equilibrium solutions f_0 to the collisionless Boltzmann equation can be constructed by specifying more or less arbitrary, albeit normalizable and non-negative, functions of the energy E and other single-valued constants of the motion. Given any such f_0, one can compute quantities such as the number density, energy density, and pressure as integrals of the distribution function. However, this is an intrinsically nonlinear procedure since the integrals themselves involve a knowledge of Φ (or g_{ab}), a quantity related to the mass density ρ (or stress energy T_a^b) by the gravitational field equation.

Both for this reason and to facilitate direct comparison with observed luminosity and/or velocity distributions, it is also useful to work the other way around. Specifically, given such moments of the distribution as the number density or pressure, one would like to determine the form (or forms) of the equilibrium distribution f_0. This is especially important given the recognition that, in general, there is no guarantee that any given ρ or T_a^b can be generated from a distribution function f_0 which is everywhere non-negative or that such an f_0, even if it exists, is unique.

It has long been known [3] that, for spherical Newtonian systems characterized by an isotropic distribution of velocities (or momenta), there exists a simple prescription to pass from the density $\rho(r)$ to a uniquely determined equilibrium f_0, although this f_0 is not always guaranteed to be positive semidefinite. The assumptions of spherical symmetry and isotropy imply that, generically, the (mass-averaged) distribution function will depend on only one constant, namely, the conserved energy E. The crucial point then is that the defining relation for $\rho(r)$ as an integral of $f_0(E)$ is easily converted to an Abel equation which can be inverted trivially. The explicit form of $f_0(E)$ is reduced thereby to a quadrature, provided that the radial coordinate r can be written explicitly as a single-valued function of Φ, so that the mass density and pressure can be viewed as functions of Φ, i.e., $\rho=\rho(\Phi)$ and $P=P(\Phi)$.

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As demonstrated by Fackerell [4], the same game can also be played in general relativity. Provided that the system is static and spherically symmetric, and that, as measured in every local orthonormal frame, the distribution of three-momenta is isotropic, one can infer generically that the distribution f_0 is again given as a function of a single constant E. In this case, E denotes the conserved energy associated with the time translation symmetry.

If the highly restrictive assumptions of spherical symmetry and isotropy are relaxed, the analysis becomes much harder since, in general, one will be dealing with equilibria that admit two or more constants of the motion. However, despite these difficulties, over the years a number of different generalizations have been formulated appropriate for Newtonian systems [5]. In particular, Hunter and Qian [6] have recently developed a powerful technique, involving a double Mellin transform, which allows one to deal with a large class of axisymmetric equilibria. This involves passing between an equilibrium $f_0(E,J_z)$, depending on both the energy E and the azimuthal angular momentum J_z, and an axisymmetric density $\rho(\varpi,z)$, where z denotes distance along the symmetry axis and $\varpi=(x^2+y^2)^{1/2}$. The only significant restriction on the allowed equilibria is that the partial derivatives $\partial\Phi/\partial z$ and $\partial\Phi/\partial\varpi$ both be monotonic, so that the density ρ can be viewed as a function of Φ and ϖ, i.e., $\rho=\rho(\Phi,\varpi)$. More recently, Qian and Hunter [7] have shown that their analysis of axisymmetric systems can also be adapted to anisotropic spherical equilibria. These systems involve equilibrium distributions f_0 depending on two conserved quantities, namely, the energy E and the square of the total angular momentum, J^2.

A generalization of the Abel transform technique to anisotropic spherical equilibria is important in view of the theoretical expectation that many systems of stars, even those which are seemingly well approximated as spherically symmetric, may be characterized by significantly anisotropic velocity distributions. This is, for example, the case for dense stellar systems such as globular clusters or galactic cusps which have undergone significant two-body relaxation: Numerical Fokker-Planck integrations indicate that, as a result of this relaxation, a nearly isotropic stellar system could develop a velocity distribution which is significantly anisotropic [8]. However, this suggests that if, as has been argued by some [9], dense relativistic systems evolved to their current state as a result of a gravothermal catastrophe triggered by binary relaxation [10], they should be characterized by velocity distributions that are far from isotropic. Two integral distributions could prove especially important for systems such as galactic nuclei containing supermassive black holes, since there is substantial evidence from N-body simulations [11] that the centers of systems containing such black holes can evolve significantly anisotropic velocity distributions.

The importance of two-integral models has increased recently because of high resolution observations of galaxies, provided by the Hubble Space Telescope [12], which permit the possibility of much more detailed modeling than had been feasible in the past, especially for the dense innermost regions which could contain supermassive black holes. Using nonparametric techniques [13], it seems possible to fit almost all the data to isotropic spherical equilibria. However, it is clearly important to investigate whether anisotropic models can provide equally good, or yet better, fits.

The objective of this paper is to demonstrate that the Newtonian algorithm of Qian and Hunter can be reformulated in a relativistic context to facilitate the construction of anisotropic distribution functions, $f_0(E,J^2)$, for specified forms of the stress-energy tensor. Section II exhibits the basic equations of relativistic stellar dynamics in a form appropriate for a spherical system characterized by an anisotropic distribution of velocities, and then shows how, in the isotropic limit, it is easy to express $f_0(E)$ as an integral of the energy density or the pressure. Section III shows that, as for the Newtonian case, one can derive an integral equation relating $f_0(E,J^2)$ to a spherically symmetric density which is viewed abstractly as a function of two variables, namely, the radial coordinate r and a metric function e^{ν}, the latter replacing the Newtonian potential Φ.

II. THE BASIC EQUATIONS OF STRUCTURE

The objects of interest here are equilibrium models for relativistic systems of nearly point mass objects, characterized as time-independent, spherically symmetric solutions to the collisionless Boltzmann equation, i.e., the Vlasov-Einstein system. These solutions involve a one-particle distribution function $f_*(x^\alpha,p_\alpha)$ defined covariantly in the cotangent bundle associated with the space-time manifold. The evolution of f_* is determined by conservation of probability, which says that f_* is conserved under transport along geodesics in a space-time metric $g_{ab}(x^c)$, the form of which is determined self-consistently from the stress-energy tensor T_a^b associated with f_* itself. Thus, explicitly, assuming geometric units with $G=c^2=1$,

$$\frac{\partial}{\partial x^a}\left[\frac{p^a}{m}f_*\right]-\frac{\partial}{\partial p_a}\left[\frac{p_b p_c}{2m}\frac{\partial g^{bc}}{\partial x^a}f_*\right]=0\ , \tag{1}$$

where the metric $g_{ab}(x^c)$ is determined by the Einstein equation

$$G_a^b[g]=8\pi T_a^b\ , \tag{2}$$

with

$$T_a^b(x^c)=\int(-g)^{-1/2}d^4p\frac{p_a p^b}{m}f_*\ . \tag{3}$$

In Eq. (3), the integral is effected in the local cotangent space (i.e., momentum space) at the space-time point x^c. Equation (1) can be viewed as deriving from the Hamiltonian $H=\sqrt{g^{ab}p_a p_b}$, a conserved quantity which is equal numerically to the particle mass m [14]. In deriving and interpreting Eq. (1), one must view x^a and p_a as the basic variables, so that, e.g., $p^b=g^{ab}p_a$ and $m=\sqrt{g^{ab}p_a p_b}$.

For notational simplicity, it will be assumed in the following that all the particles being described have the same rest mass, so that

$$f_*(x^a,p_a)=f(x^a,p_a)\delta_D(m-m_*) \ , \tag{4}$$

where δ_D denotes a Dirac delta restricting the support of f_* to the mass shell hyperboloid. This entails no loss of generality since all the manipulations performed below can be effected equally well for a mass-averaged distribution function if more than one mass species is assumed to exist. The case of an isotropic distribution of three-momenta with unequal masses was considered explicitly in [4]. For further computational simplicity, it will also be assumed that units are so chosen that the rest mass $m=1$.

Given the assumption of a spherical equilibrium, it is convenient to introduce Schwarzschild coordinates [15]. This leads to a diagonal line element of the form

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2) \ , \tag{5}$$

where the metric functions e^{ν} and e^{λ} are functions of r, independent of the coordinate time t. One convenient feature about this choice of coordinates is that the stress energy associated with the equilibrium f_0 is necessarily diagonal. The assumption of a spherically symmetric equilibrium implies generically that the distribution function f_0 will be given as a function of two conserved quantities (so-called isolating integrals), namely, E and J^2. Here $E=p_t$ is the conserved energy associated with time translation symmetry, i.e., the existence of a timelike Killing field. $J^2=p_\theta^2+p_\varphi^2/\sin^2\theta$ is the squared angular momentum, a conserved quantity associated with the three rotational Killing fields.

The fact that $f_0=f_0(E,J^2)$ implies further that $T_\theta^\theta=T_\varphi^\varphi$, so that only three of the four nonvanishing components of T_a^b can be unequal. Similarly, it follows that the number current,

$$N_a=\int(-g)^{-1/2}d^4p\frac{p_a}{m}f_* \ , \tag{6}$$

only has one nonvanishing component, namely, N_t. Explicit computation then reveals that the energy density

$$\begin{aligned}\rho(r)&\equiv-T_t^t(r)\\&=2\pi e^{-3\nu/2}\int dJ^2\int dE\, f_0(E,J^2)KE^2r^{-2} \ ,\end{aligned} \tag{7}$$

the radial pressure

$$P_R(r)\equiv T_r^r(r)=2\pi e^{-\nu/2}\int dJ^2\int dE\, f_0(E,J^2)K^{-1}r^{-2} \ , \tag{8}$$

the tangential pressure

$$\begin{aligned}P_T(r)&\equiv T_\theta^\theta(r)\\&=T_\varphi^\varphi(r)=\pi e^{-\nu/2}\int J^2dJ^2\int dE\, f_0(E,J^2)Kr^{-4} \ ,\end{aligned} \tag{9}$$

and the number density

$$\begin{aligned}e^{\nu/2}n(r)&=-N_t(r)\\&=2\pi e^{-\nu/2}\int dJ^2\int dE\, f_0(E,J^2)KEr^{-2} \ ,\end{aligned} \tag{10}$$

where

$$K\equiv(E^2e^{-\nu}-1-r^{-2}J^2)^{-1/2} \ . \tag{11}$$

In each case, the integrations extend over intervals defined by the inequalities

$$J^2\ge 0 \ \text{ and } \ (E^2e^{-\nu}-1-r^{-2}J^2)\ge 0 \ . \tag{12}$$

The collisionless Boltzmann equation implies a purely hydrodynamic description, defined in configuration space, that can be formulated and solved without explicit reference to the phase space distribution $f_0(E,J^2)$. This description involves three different matter variables, namely, $\rho(r), P_R(r)$, and $P_T(r)$, each generated as a moment of the Vlasov equation, that are coupled to gravity by the metric function $e^{\nu(r)}$, which plays the role of the Newtonian potential Φ. The matter variables, which are related via some generalized equation of state, satisfy a relativistic analog of the Euler equation, which can be derived by analogy with the Jeans equation of Newtonian galactic dynamics. The metric function $e^{\nu(r)}$ or, equivalently, $\nu(r)$, satisfies an anisotropic generalization of the Oppenheimer-Volkoff equation which itself generalizes the Newtonian relation $d\Phi/dr=-m(r)/r^2$. The remaining metric function $e^{\lambda(r)}$ is better viewed as an auxiliary variable which is readily eliminated from the structure equations.

Geometrically, the desired Jeans equation corresponds to the radial component of the relation

$$q_{bc}\nabla_a T^{ab}=0 \ , \tag{13}$$

with

$$q_{ab}=g_{ab}-u_au_b \ , \tag{14}$$

which yields the projection of the energy conservation equation orthogonal to the flow associated with the four-velocity u^a. More physically, perhaps, it can be derived directly from the Vlasov equation: Multiply Eq. (1) by p_a and then perform a momentum space integration d^4p. Assuming appropriate falloff conditions for large momenta, the second term can be integrated by parts with the vanishing surface term. By both multiplying and dividing by $\sqrt{-g}$ one then infers that

$$\frac{\partial}{\partial x^a}\sqrt{-g}\,T_b^a+\frac{1}{2}\frac{\partial g^{cd}}{\partial x^b}\sqrt{-g}\,T_{cd}=0 \ , \tag{15}$$

with T_a^b given by Eq. (3).

The angular components of this equation lead to the trivial identity $0=0$, a consequence of the assumption of a spherically symmetric configuration. Similarly, the temporal component is trivial because of the assumption of time translation symmetry. However, the radial component does have nontrivial content. Specifically, by evaluating the derivatives of the metric functions and replacing T_a^b by the appropriate matter variables, one obtains the desired Jeans equation,

$$\frac{dP_R}{dr}+\frac{2}{r}(P_R-P_T)=-\frac{1}{2}(\rho+P_R)\frac{d\nu}{dr} \ . \tag{16}$$

The desired source equation for $d\nu/dr$ follows immediately from the G_t^t and G_r^r field equations [15]. The G_t^t equation, which expresses T_t^t in terms of λ and r, can be solved to give

$$e^{-\lambda}=1-\frac{2m}{r} , \tag{17}$$

where

$$m(r)=4\pi\int dr\, r^2\rho(r) . \tag{18}$$

But, by combining Eq. (18) with the G_r^r equation, which expresses T_r^r in terms of λ, ν, and r, one is led to the desired relation

$$\frac{d\nu}{dr}=\frac{2(m+4\pi P_R r^3)}{r(r-2m)} . \tag{19}$$

The other nonvanishing components of the Einstein equation, G_θ^θ and G_φ^φ, are redundant. As is well known [15], they follow from the other field equations by virtue of the Bianchi identity, but the Bianchi identity is equivalent to the energy conservation equation used in deriving Eq. (16).

Considering only the hydrodynamic variables, one is confronted with a system of two equations—the Jeans equation and the source equation—for four unknowns, namely ρ, P_R, P_T, and ν. It is thus clear that, without imposing any additional restrictions, there is an enormous amount of freedom in the construction of solutions. Indeed, what is required is some generalization of an equation of state to obtain a closure condition.

For the special case of a system characterized at each point in space by an isotropic distribution of three-velocities, the distribution function f_0 can depend only on the energy E. It follows that the pressure tensor must also be isotropic, with $P_R=P_T\equiv P$. In a hydrodynamical description, one is thus reduced to three unknowns for two equations, but some freedom still remains. It therefore seems reasonable to impose an additional condition, e.g., by specifying ρ explicitly as a function of r.

Given some $\rho(r)$, one is reduced finally to a system of two equations in two unknowns, for which, at least in principle, a general solution can be derived. The only question is whether this solution set $\{\rho,P,\nu\}$ is consistent with an acceptable distribution function f_0, i.e., whether the given $\rho(r)$ can be generated from some isotropic $f_0(E)$. Since such a construction would involve a mapping between one function of one variable and another function of one variable, one might expect (1) that an $f_0(E)$ does exist and (2) that this $f_0(E)$ is unique, although there would be no reason *a priori* to expect that this $f_0(E)$ is physically acceptable, e.g., positive semidefinite.

In fact this expectation is true: Assuming that e^ν and e^λ are both strictly positive, it follows from Eqs. (17) and (19) that $de^\nu/dr>0$, so that e^ν is a monotonic function of r. However, given that e^ν is monotonic in r, so that ρ and P can be viewed as functions of e^ν, one can manipulate the equations for $\rho(r)$ or $P(r)$ to yield $f_0(E)$ as an integral over the specified matter variable. To see this, define rescaled distribution functions

$$\mathcal{F}_\rho(\xi)=2\pi\xi^{1/2}f_0(\xi^{1/2}) ,$$
$$\mathcal{F}_P(\xi)=\tfrac{2}{3}\pi\xi^{-1/2}f_0(\xi^{1/2}) , \tag{20}$$

and rescaled matter variables

$$G_\rho(b)=b^2\rho(b) \quad\text{and}\quad G_P(b)=b^2P(b) , \tag{21}$$

where

$$\xi=E^2 \quad\text{and}\quad b=e^\nu . \tag{22}$$

With these redefinitions, it follows trivially that the integral equations for ρ and P can be rewritten as

$$G_\rho(b)=\int_b^1\mathcal{F}_\rho(\xi)(\xi-b)^{1/2}d\xi ,$$
$$G_P(b)=\int_b^1\mathcal{F}_P(\xi)(\xi-b)^{3/2}d\xi . \tag{23}$$

Suppose now that the system of interest has a smooth outer boundary, so that surface terms can be neglected with impurity. In this case, one can differentiate the ρ equation once and the P equation twice to obtain

$$\int_b^1\mathcal{F}_\rho(\xi)(\xi-b)^{-1/2}d\xi=-\frac{1}{2}\frac{dG_\rho}{db}\equiv\mathcal{J}_\rho(b)$$

and

$$\int_b^1\mathcal{F}_P(\xi)(\xi-b)^{-1/2}d\xi=\frac{4}{3}\frac{d^2G_P}{db^2}\equiv\mathcal{J}_P(b) . \tag{24}$$

However, each of these is an Abel integral equation for $\mathcal{F}_i$, which, neglecting surface terms, admits a solution of the form

$$\mathcal{F}_i(\xi)=\frac{1}{\pi}\frac{d}{d\xi}\int_\xi^b db\,\mathcal{J}_i(b)(b-\xi)^{-1/2} , \tag{25}$$

with $i=\rho$ or P. The transform in terms of P was first formulated by Fackerell [4]. However, the ρ transform is the more direct generalization of the transform of the Newtonian mass density originally identified by Eddington [3].

For an anisotropic pressure tensor, the situation is more complicated. In this case, one has four unknowns but only two equations, so that two conditions are required to get a closed hydrodynamic system. The obvious question, then, is whether the closure conditions are consistent with a distribution function $f_0(E,J^2)$. The key point here is that, since f_0 now depends on two variables, it would seem reasonable to expect that one can in fact impose two nontrivial restrictions on the moments of f_0. The procedure described in Sec. III, based on the work of Qian and Hunter [6,7], shows that this is in fact the case.

III. THE INTEGRAL TRANSFORM

Since the physical density ρ is a function of only one coordinate, namely, r, it is not reasonable to expect that it can be used by itself to compute a uniquely determined two-integral $f_0(E,J^2)$. Rather, to obtain an integral equation for f_0, one really needs somehow to specify a function of two variables. This was first done Newtonianly by Dejonghe [16] through the introduction of a so-called augmented density, using a prescription which admits a completely straightforward relativistic generalization.

In the context of general relativity, the starting point is the definition of augmented densities and pressures, $\rho(b,r)$, $P_R(b,r)$, and $P_T(b,r)$ via the integrals (7)–(9), *assuming that the hydrodynamic equations have not been imposed, so that* $b=e^{\nu}$ *is not yet a known function of* r. Once the hydrodynamic relations are imposed, these augmented quantities reduce to functions of a single variable, which can be taken to be either b or r, or some combination thereof, but in general they define perfectly acceptable functions of two independent variables.

One crucial feature about these augmented quantities is that they are related to one another by simple differential equations. Thus, in particular, it is easily verified that the radial and tangential pressures can be generated explicitly from a knowledge of the energy density ρ. Indeed, differentiation of Eqs. (7)–(9) leads to the simple relations

$$b^{1/2}\frac{\partial}{\partial b}b^{1/2}P_R(b,r)=-\frac{1}{2}\rho(b,r^2) \tag{26}$$

and

$$b^{1/2}\frac{\partial}{\partial b}b^{1/2}P_T(b,r)=-\frac{1}{2}\frac{\partial}{\partial r^2}r^2\rho(b,r^2)\ . \tag{27}$$

Note parenthetically that this implies a simple expression for the difference between the radial and tangential pressures, namely,

$$b^{1/2}\frac{\partial}{\partial b}b^{1/2}[P_R(b,r^2)-P_T(b,r^2)]=\frac{r^2}{2}\frac{\partial}{\partial r^2}\rho(b,r^2)\ . \tag{28}$$

The specific objective of the remainder of this section is to derive an expression for $f_0(E,J^2)$ in terms of a contour integral involving the augmented density $\rho(b,r^2)$. Different choice of $\rho(b,r^2)$ corresponding to the same physical density will in general yield different two-integral distributions $f_0(E,J^2)$, this reflecting the fact that more than one $f_0(E,J^2)$ corresponds to the same physical $\rho(r)$. However, Eqs. (26) and (27) imply that different chains of augmented density correspond to different choices of radial and tangential pressure, so that the different distributions $f_0(E,J^2)$ correspond simply to different choices of $P_R(r)$ and $P_T(r)$.

If the limits of integration are written explicitly in the integrals of Eq. (7), it follows that, in terms of the variables ξ and J^2,

$$\rho(b,r^2)=\pi b^{-3/2}\int_b^1 d\xi\int_0^{r^2(\xi/b-1)}dJ^2\frac{\xi^{1/2}f_0(\xi,J^2)}{r\sqrt{r^2(\xi/b-1)-J^2}}. \tag{29}$$

Now introduce a new, transformed density $\bar{\rho}$ satisfying

$$\bar{\rho}(b,R^2)=\frac{1}{\pi}\int_0^{R^2}dr^2\frac{\rho(b,r^2)}{\sqrt{r^2(R^2-r^2)}}\ . \tag{30}$$

If one inserts $\rho(b,r^2)$ into Eq. (30) and rearranges the limits of integration in the (J^2,r^2) plane, the dr^2 integration can be performed explicitly, leading to the relatively simple expression

$$b^{3/2}\bar{\rho}(b,R^2)=\frac{\pi}{R}\int_b^1 d\xi\int_0^{R^2(\xi/b-1)}dJ^2\frac{\xi^{1/2}f_0(\xi,J^2)}{\sqrt{J^2}}\ . \tag{31}$$

This relation can be simplified yet further through the identification of new radial coordinates satisfying

$$s^2=b^{-1}r^2 \quad\text{and}\quad S^2=b^{-1}R^2\ . \tag{32}$$

For s^2, and hence S^2, to serve as a satisfactory radial coordinate, it must be true that the derivative $ds^2/dr^2>0$, so that the mapping from r^2 to s^2 is one to one. This is in fact a nontrivial condition, equivalent to

$$1-\frac{r}{2}\frac{d\nu}{dr}>0\ , \tag{33}$$

or, from Eq. (19),

$$r>3m(r)+4\pi P_R r^2\ . \tag{34}$$

However, this restriction does not seem unreasonable, given the recognition that, at a radius r for which $(r/2)(d\nu/dr)=1$, the gravitational field is so strong that a circular orbit will have an infinite energy, i.e., $E^2\to\infty$.

That this is the case is easy to see. The mass shell constraint implies that, for an orbit with $dr/d\tau=0$,

$$E^2=e^{\nu}\left[1+\frac{J^2}{r^2}\right]\equiv V(r)\ , \tag{35}$$

but the condition $\partial V/\partial r=0$ then yields

$$E^2=e^{\nu}\Big/\left[1-\frac{r}{2}\frac{d\nu}{dr}\right]\ . \tag{36}$$

With the introduction of the new coordinate S^2, the augmented density $\bar{\rho}$ is to be viewed as a function of b and S^2, so that Eq. (31) takes the form

$$b^2\bar{\rho}(b,S^2)\equiv G(b,S^2)=\frac{\pi}{S}\int_b^1 d\xi\int_0^{S^2(\xi-b)}dJ^2\frac{\xi^{1/2}f_0(\xi,J^2)}{\sqrt{J^2}}\ . \tag{37}$$

Differentiation with respect to b then yields a simple, single-integral equation, namely,

$$\frac{\partial G(b,S^2)}{\partial b}=-\pi\int_b^1 d\xi\frac{\xi^{1/2}f_0(\xi,S^2(\xi-b))}{\sqrt{\xi-b}}\ . \tag{38}$$

This expression is very similar to the Newtonian relation obtained by Qian and Hunter [7]. Indeed, by writing

$$b=1+2\Phi,\quad \xi=1+2\mathcal{E},\quad F(\mathcal{E},J^2)=\xi^{1/2}f_0(\xi,J^2)\ , \tag{39}$$

where $\mathcal{E}$ and Φ are the new independent variables and F is the rescaled distribution function, one finds that Eq.

(38) reduces to

$$\frac{\partial G(\Phi,S^2)}{\partial\Phi}=-2\sqrt{2}\pi\int_{\Phi}^{0}d\mathcal{E}\frac{F(\mathcal{E},S^2(\mathcal{E}-b))}{\sqrt{\mathcal{E}-\Phi}} . \quad (40)$$

This is essentially the same as the integral equation formulated by Qian and Hunter, the only differences being (1) that their basic variables were the relative energy and potential, $-\mathcal{E}$ and $-\Phi$, and (2) that they close to introduce explicitly the value of the potential at the edge of the matter configuration.

Noting that the relativistic description introduces no new poles or branch cuts, an analysis analogous to that presented in Hunter and Qian [6] can thus be used to invert Eq. (40). Thus, in particular, if the augmented density $G(b,S^2)$ is analytic in b in a suitable region in the complex plane, one infers that $F(E,J^2)$ can be expressed as a contour integral of the form

$$F(\mathcal{E},J^2)=\frac{1}{4\pi^2 i\sqrt{2}}\frac{\partial}{\partial\mathcal{E}}\int_{\Phi_-}^{0}\frac{d\Phi}{\sqrt{\mathcal{E}-\Phi}}G_1\left[\Phi,\frac{J^2}{2(\mathcal{E}-\Phi)}\right] . \quad (41)$$

In this equation, the integration contour corresponds to a loop in the complex Φ plane running along the real Φ axis to the right of the branch point at $\Phi=\mathcal{E}$. The quantity G_1 denotes a derivative of G with respect to the first argument, i.e., neglecting the dependence on Φ arising because the second argument also has some Φ dependence. If one restores the "natural" variables ξ and b, this becomes instead

$$F(\xi,J^2)=\frac{1}{2\pi^2 i}\frac{\partial}{\partial\xi}\int_{b_-}^{1}\frac{db}{\sqrt{\xi-b}}G_1\left[b,\frac{J^2}{(\xi-b)}\right] , \quad (42)$$

where, explicitly,

$$F(\xi,J^2)=\xi^{1/2}f_0(\xi,J^2) . \quad (43)$$

Here the contour is a loop in the complex b plane running along the real axis to the right of the branch point at $b=\xi$.

Translating from S^2 back to the old radial coordinate R^2, and thus viewing G as a function of b and R^2, Eq. (38) takes the form

$$\frac{\partial G(b,R^2)}{\partial b}+\frac{R^2}{b}\frac{\partial G(b,R^2)}{\partial R^2}=-\pi\int_{b}^{1}d\xi\frac{F(\xi,J^2)}{\sqrt{\xi-b}} , \quad (44)$$

and the final integral relation becomes

$$F(\mathcal{E},J^2)=\frac{1}{2\pi^2 i}\frac{\partial}{\partial\xi}\int_{b_-}^{1}\frac{db}{\sqrt{\xi-b}}\left[G_1\left[b,\frac{bJ^2}{(\xi-b)}\right]+\frac{J^2}{(\xi-b)}G_2\left[b,\frac{bJ^2}{(\xi-b)}\right]\right] , \quad (45)$$

where, now, G_1 and G_2 denote, respectively, derivatives with respect to the first and second arguments.

The final relation (45) does not involve the derivative $\partial S^2/\partial R^2$ explicitly, so that one might anticipate that it remains valid even if S^2 is not a monotonic function of R^2. This may in fact be true. However, the analyticity properties of $\bar{\rho}(b,R^2)$ and $\bar{\rho}(b,S^2)$ differ when S^2 is not monotonic, and any justification of Eq. (45) will depend upon the details of precisely how the analyticity properties of $\bar{\rho}$ are changed.

The isotropic limit of the two-integral formalism described here is easily understood. If one imposes the constraint that the augmented radial and tangential pressures are equal, it follows from Eq. (28) that the augmented density $\rho(b,r^2)$ is in fact a function only of b. Consistent with Eqs. (26) and (27) one can then assume that the augmented P_R and P_T also depend only on b. Given, however, that ρ, P_R, and P_T are functions only of b, one is reduced immediately to the isotropic case, with the pressure $P_R=P_T=P$ satisfying

$$b\frac{dP}{db}=-\frac{1}{2}(\rho+P) , \quad (46)$$

which is equivalent to the isotropic version of the Jeans equation (16).

It is also possible to choose augmented pressures P_R and P_T which, albeit unequal, have the property that they agree when b is viewed as a function of r^2, i.e.,

$$P_R(b,r^2)\neq P_T(b,r^2) \text{ but } P_R(r)=P_T(r) . \quad (47)$$

Such a choice of augmented pressures corresponds to an augmented ρ which exhibits a nontrivial dependence on both b and r^2 and thus, according to Eq. (45), implies an anisotropic two-integral $f_0(E,J^2)$. In other words, at least in principle one can have an anisotropic distribution $f_0(E,J^2)$ which leads at each point in space to an isotropic pressure tensor, with $P_R=P_T=P$.

One final point should be made. To facilitate direct contact with observations one might like to formulate an analog of Eq. (45) involving the number density n, rather than the energy density ρ. This, however, seems problematic. The hydrodynamic equations provide concrete relations among the quantities ρ, P_R, P_T, and ν, so that, given a knowledge of $\rho(r)$ and some constraint connecting $P_R(r)$ and $P_T(r)$, one can determine $\nu(r)$ and the explicit form of $P_R(r)$ and $P_T(r)$ without solving for $f_0(E,J^2)$. However, a knowledge of these matter variables is not sufficient to determine the form of the number density n. Only after solving for $f_0(E,J^2)$ can $n(r)$ be computed explicitly. Related to this is the fact that, although one can define an augmented density $n(b,r^2)$, this augmented n cannot be related to the augmented ρ, P_R, and P_T via any simple analog of Eqs. (26) and (27).

In principle, one could perhaps specify $n(r)$ and $\nu(r)$ as given functions and then proceed to define an augmented density $n(b,r^2)$ which could be manipulated to derive an analog of Eq. (45). However, the augmented $n(b,r^2)$ is not easily related to the other matter variables, so that the assumed splitting into b and r^2 is not constrained in any obvious physical way.

ACKNOWLEDGMENTS

H.E.K. was supported in part by the National Science Foundation Grant No. PHY92-03333. Some of this work was done while H.E.K. was a visitor at the Aspen Center for Physics, the hospitality of which is acknowledged gratefully.

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